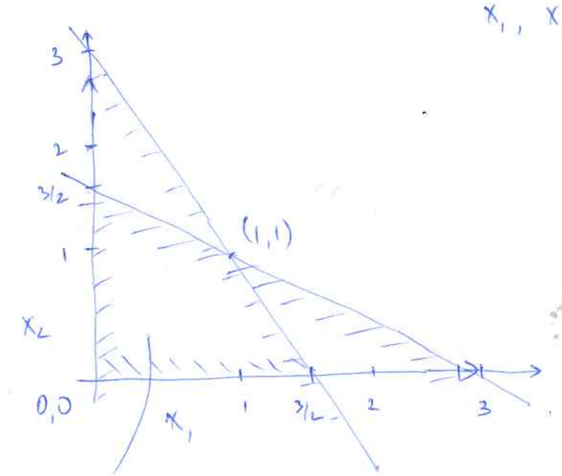
Linear Programming: Lecture 20.

Problem of minimizing a <u>linear</u>, multivariable objective, subject to <u>linear</u> constraints

Eg.: LP1:

$$\text{maximize} \quad x_1 + x_2 \qquad \Big] \text{ objective.}$$

$$\begin{aligned} \text{subject to:} \quad x_1 + 2x_2 &\leq 3 \\ 2x_1 + x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned} \quad \Bigg] \text{ constraints}$$



feasible region.

fix c, consider line $x_1 + x_2 = c$. (slope -1, x_2-intercept at c)

if there exists feasible point on this line, then $\exists$ feasible soln w/ objective value c.

Hence, our goal is to max. c s.t. $\exists$ feasible x_1, x_2 on the line $x_1 + x_2 = c$

this happens at the "corner" $(1,1)$

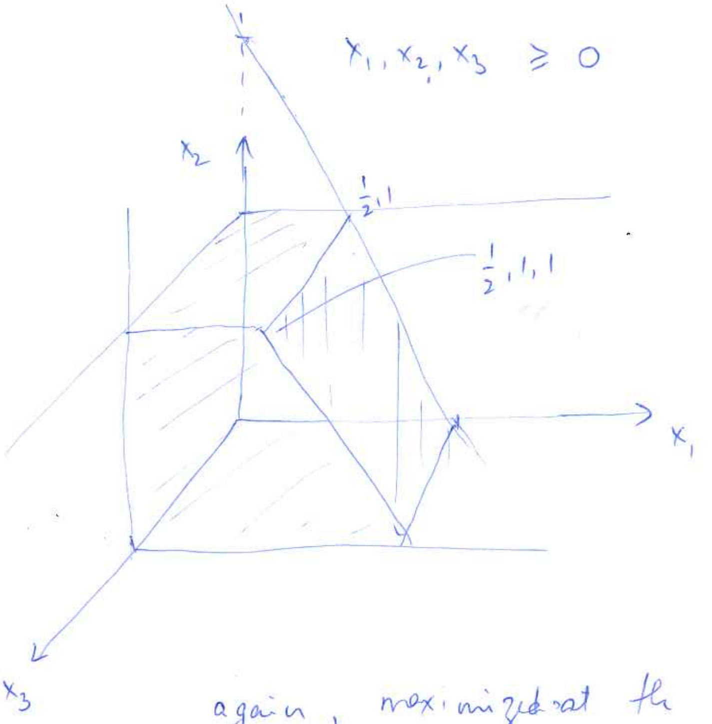
"Corners" are v.v.v. impt. in linear programming; this is where all the action happens!

- dimension of an LP is (generally) the # of variables.

LP2:

$$\max \quad x_1 + x_2 + x_3$$

$$\begin{aligned} \text{s.t.} \quad 2x_1 + x_2 &\leq 2 \\ x_2 &\leq 1 \\ x_3 &\leq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$



again, maximized at the "corner" $(\frac{1}{2}, 1, 1)$

more generally: an LP is of the form:

$$\min \quad c^T x \qquad\qquad x = (x_1, \ldots, x_n)$$

$$\begin{aligned} \text{s.t.} \quad a_i^T x &\geq b_i & i &\in M_1 \\ a_i^T x &\leq b_i & i &\in M_2 \\ a_i^T x &= b_i & i &\in M_3 \\ x_j &\geq 0 & j &\in N_1 \\ x_j &\leq 0 & j &\in N_2 \end{aligned}$$

M_1, M_2, M_3, N_1, N_2 are index sets.

we will stick to LPs of the form

$$\min \; c^T x$$

$$\text{s.t.} \quad a_i^T x \geq b_i \qquad i \in [m]$$

$$x \geq 0$$

this is w/o loss of generality, since:

- $a_i^T x \leq b_i \equiv -a_i^T x \geq -b_i$
- $a_i^T x = b_i \equiv a_i^T x \geq b_i, \; a_i^T x \leq b_i$
- if $x_j \leq 0$, replace x_j by $-x_j$, $x_j \geq 0$
- if x_j is "free", i.e., neither $x_j \geq 0$ nor $x_j \leq 0$, replace x_j by $x_j^+ - x_j^-$, and $x_j^+, x_j^- \geq 0$

or, equivalently:

$$\min \; c^T x$$

$$\text{s.t.} \quad Ax \geq b \qquad - \; m \text{ constraints, } n \text{ variables, } A \in \mathbb{R}^{m \times n}$$

$$x \geq 0$$

"Solving" a linear program: Some geometry.

Given $a \in \mathbb{R}^n$, $b \in \mathbb{R}$,

- the set $\{x \in \mathbb{R}^n : a^T x = b\}$ is a hyperplane
- the set $\{x \in \mathbb{R}^n : a^T x \leq b\}$ is a half-space.
- a polyhedron is the intersection of a finite set of halfspaces:

$$P = \{x : Ax \leq b\}$$

- a bounded polyhedron is called a <u>polytope</u>.

(thus, the feasible region of any LP is a polyhedron).

now let's try & understand "corners". Let $P = \{x : Ax \leq b\}$ be a polyhedron.

[~~a cone~~]

- A constraint $a_i^T x \leq b_i$ is <u>tight/active/binding</u> at $x^* \in P$ if $a_i^T x^* = b_i$
- constraints $a_i^T x \leq b_i$, $a_j^T x \leq b_j$ are <u>linearly independent</u> if vectors a_i, a_j are linearly independent. (can extend to more than 2 constraints in the usual way)

<u>Defn</u>: $x^* \in P$ is a <u>basic solution</u> if n l.i. constraints are tight at x^*.

(note: x^* is the unique pt. in $\mathbb{R}^n$ where these n constraints are tight).

$x^* \in P$ is a <u>basic feasible solution</u> if x^* is basic, and $Ax^* \leq b$, i.e., x^* is feasible.

(note: # of basic solutions $\leq \binom{m}{n}$)

→ Now here is why corners/bfs's are important.

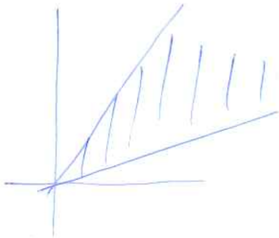
<u>Theorem</u>: Suppose polyhedron P is nonempty, and has at least one bfs. Consider the objective of minimizing $c^T x$. Then either optimal value is $-\infty$, or there is an optimal bfs.

<u>Theorem</u>: If polyhedron P ~~is bounded~~ does not contain a line, it contains at least one bfs.

(hence if P is bounded, it contains at least one bfs)

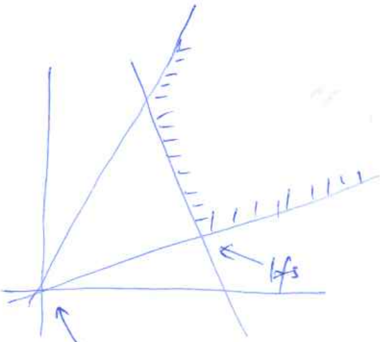
Some examples:

- unbounded polyhedron:



note that a polyhedron of the form $P = \{x : Ax \geq b, x \geq 0\}$ does not contain a line, and hence contains at least one bfs.

- basic solution that is not feasible:



also unbounded

~~[Now here is why bfs's / corners are important]~~

For polytopes, we then have a simple algorithm for finding the optimal solution (i.e, solving an LP):

1. enumerate all basic solutions ($\binom{m}{n}$ possible solutions)
2. check if each is feasible
3. choose bfs that minimizes $c^T x$.

Let's consider some of the problems we've seen, and try & write them as LPs.

① maximum matchings (given undirected graph $G = (V, E)$)

say variable x_e for each $e \in E$, $x_e = 1 \Rightarrow e$ in matching
$= 0$ o.w.

$$\max \sum_{e \in E} x_e$$

$$\forall v, \quad \sum_{e \text{ incident to } v} x_e \leq 1$$

$$x_e \geq 0$$

$$x_e \in \{0,1\}$$ ← integer constraints, give an integer linear program (ILP)

convince yourself that:

(i) M is a matching iff $\{x : x_e = 1 \text{ if } e \in M, = 0 \text{ o.w.}\}$ is a feasible solution.

(ii) optimal solution corresponds to maximum matching.

② maximum ~~f~~ s-t flow, ~~capacity~~ (given directed graph $G = (V, E)$, vertices s, t, capacity c_e on edges).

Variable x_e for each edge, equal to amt. of flow on edge

$$\max \sum_{e \text{ out of } s} x_e$$

$$\forall v \neq s,t \quad \sum_{e \text{ into } v} x_e - \sum_{e \text{ out of } v} x_e = 0$$

$$\forall e \quad x_e \leq c_e$$

$$\forall e \quad x_e \geq 0$$